

# One-way deficit and quantum phase transitions detection in $XY$ model

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Quantum deficit originates in questions regarding work extraction from quantum systems coupled to a heat bath [Phys. Rev. Lett. **89**, 180402 (2002)]. It is a kind of quantum correlations besides entanglement and quantum discord, and links quantum thermodynamics with quantum correlations. In this paper, we evaluate the one-way deficit of two adjacent spins in the bulk for the  $XY$  model. We find that the one-way deficit characterizes the quantum phase transition in the  $XX$  model and the transverse field Ising model that the  $XY$  model reduces to for specified parameters. This study may enlighten extensive studies of quantum phase transitions from the perspective of quantum information and quantum correlations, including finite-temperature phase transitions, phase transitions of other quantum many-body models or even topological phase transitions.

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## I. INTRODUCTION

Quantum deficit is a kind of nonclassical correlation besides entanglement and quantum discord. Quantum deficit [1–3] originates on asking how to use nonlocal operation to extract work from a correlated system coupled to a heat bath only in the case of pure states [1]. In the general case, the advantage is related to more general forms of quantum correlations. Oppenheim *et al.* defined the work deficit [1] is a measure of the difference between the information of the whole system and the localizable information [4, 5]. Recently, Streltsov *et al.* [6, 7] give the definition of the one-way information deficit by means of relative entropy, which is also called one-way deficit that uncovers an important role of quantum deficit as a resource for the distribution of entanglement.

Many developments in quantum information processing [8] has provided much insight into quantum phase transitions [9]. Especially, quantum correlations has been successful in characterizing a large number of critical phenomena of great interest. In particular, entanglement was the first and most outstanding member to detect a number of critical points, see [9–14]. Furthermore, quantum discord is an outstanding quantum correlation, is also used to study quantum phase transitions [15]. Another indication of quantumness that is also found its applications for probing quantum phases and quantum phase transition [16–20].

In this paper, we analytically calculate the one-way deficit of the thermal ground state of two adjacent spins in the bulk of the  $XY$  model. We find that the one-way deficit may characterize the quantum phase transition in the  $XX$  model and the transverse field Ising model that the  $XY$  model reduces to for specified parameters. In details, we find the phenomenon of sudden death of the one-way deficit in  $XX$  model as the strength of transverse field increases to the critical point; the one-way

deficit of nearby two spins arrives its maximal value almost at the critical point of transverse field Ising model. Our results will enlighten extensive studies of the quantum information properties of ground states in different phases of critical systems. On the contrary, it will also benefit many applications such as to detect the quantum phase transition and to evaluate the capacity of quantum computations in critical systems.

## II. ONE-WAY DEFICIT IN $XY$ MODEL

One-way deficit by von Neumann measurement on one side is given by [21]

$$\Delta^{\rightarrow}(\rho^{ab}) = \min_{\{\Pi_k\}} S\left(\sum_k \Pi_k \rho^{ab} \Pi_k\right) - S(\rho^{ab}), \quad (1)$$

where  $S(\cdot)$  denotes to the von Neumann entropy. As a kind of quantum correlations besides entanglement and quantum discord, one-way deficit links quantum thermodynamics with quantum correlations and deserve further investigations in critical systems.

Then, we will consider the  $XY$  model [22] in the zero-temperature case. The Hamiltonian of our model is as follows [23]:

$$H = - \sum_{i=0}^{L-1} \left[ \frac{(1+\gamma)\sigma_1^i \sigma_1^{i+1} + (1-\gamma)\sigma_2^i \sigma_2^{i+1}}{2} + h\sigma_3^i \right] \quad (2)$$

with  $L$  being the number of spins in the chain,  $\sigma_n^i$  the  $i$ th spin Pauli operator in the direction  $n = 1, 2, 3$  and periodic boundary conditions assumed. The  $XX$  model and transverse field Ising model thus correspond to the special cases for this general class of models. For the case that  $\gamma \rightarrow 0$ , our model reduces to  $XX$  model. When  $\gamma = 1$ , the model reduces to transverse field Ising model [24]. In fact, there exists additional structure of interest in phase space beyond the breaking of phase flip symmetry at  $h = 1$ , which is the critical point between two quantum phases. It is worth noting that there exists a quarter of

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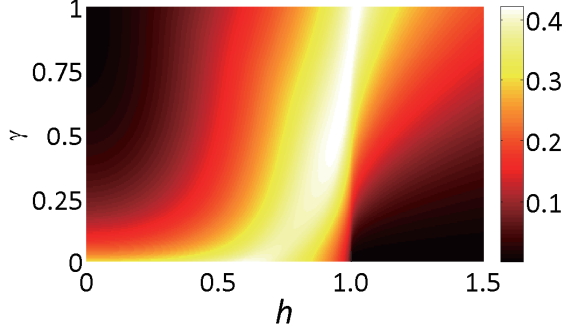


FIG. 1: (Color online) One-way deficit of two adjacent spins in the bulk for the XY model in the thermodynamic limit as a function of the quantum parameter  $h, \gamma$ .

circle,  $h^2 + \gamma^2 = 1$ , on which the ground state is fully separable.

For the thermal ground state of XY model written as Equation (2), the Bloch representation of the reduced density matrix for two nearby spins at positions  $i$  and  $i + 1$  has been obtained in [25] as

$$\rho^{ab} = \frac{1}{4}(I \otimes I + r\sigma_3 \otimes I + sI \otimes \sigma_3 + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i), \quad (3)$$

where  $I$  is the identity,  $r = s = \langle \sigma_3^i \rangle$ ,  $c_1 = \langle \sigma_1^i \sigma_1^{i+1} \rangle$ ,  $c_2 = \langle \sigma_2^i \sigma_2^{i+1} \rangle$ , and  $c_3 = \langle \sigma_3^i \sigma_3^{i+1} \rangle$ . In the thermal limit  $T \rightarrow 0$ , we have  $\langle \sigma_3^i \rangle = -\frac{1}{\pi} \int_0^\pi \frac{d\phi}{\omega_\phi} (1 + \cos \phi/h)$ ,  $\langle \sigma_3^i \sigma_3^{i+1} \rangle = \langle \sigma_3^i \rangle^2 - G_1 G_{-1}$ ,  $\langle \sigma_1^i \sigma_1^{i+1} \rangle = G_{-1}$  and  $\langle \sigma_2^i \sigma_2^{i+1} \rangle = G_1$  where we have

$$G_{\pm 1} \equiv -\frac{1}{\pi} \int_0^\pi \frac{d\phi}{\omega_\phi} [\cos(\phi)(1 + \cos \phi/h) \mp \gamma \sin(\phi) \sin(\phi)/h]. \quad (4)$$

and  $\omega_\phi = \sqrt{(\gamma h^{-1} \sin \phi)^2 + (1 + h^{-1} \cos \phi)^2}$ .

The eigenvalues of the  $X$  states in Eq. (3) are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{4}(1 - c_3 \pm |c_1 + c_2|), \\ \lambda_{3,4} &= \frac{1}{4}[1 + c_3 \pm \sqrt{(2r)^2 + (c_1 - c_2)^2}]. \end{aligned} \quad (5)$$

with which the entropy is given by

$$S(\rho) = -\sum_{i=1}^4 \lambda_i \log \lambda_i. \quad (6)$$

Next, we evaluate the one-way deficit of the  $X$  states in Eq. (3). Let  $\{\Pi_k = |k\rangle\langle k|, k = 0, 1\}$  be the local measurement for the party  $b$  along the computational base  $|k\rangle$ ; then any von Neumann measurement for the party  $b$  can be written as

$$\{B_k = V\Pi_k V^\dagger : k = 0, 1\} \quad (7)$$

for some unitary  $V \in U(2)$ . For any unitary  $V$ ,

$$V = tI + i\vec{y} \cdot \vec{\sigma} = \begin{pmatrix} t + y_3 i & y_2 + y_1 i \\ -y_2 + y_1 i & t - y_3 i \end{pmatrix}. \quad (8)$$

with  $t \in \mathbb{R}$ ,  $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ , and

$$t^2 + y_1^2 + y_2^2 + y_3^2 = 1, \quad (9)$$

after the measurement  $B_k$ , the state  $\rho^{ab}$  will be changed into the ensemble  $\{\rho_k, p_k\}$  with

$$\rho_k = \frac{1}{p_k}(I \otimes B_k)\rho(I \otimes B_k), \quad (10)$$

$$p_k = \text{tr}(I \otimes B_k)\rho(I \otimes B_k). \quad (11)$$

To evaluate  $\rho_k$  and  $p_k$ , we write

$$\begin{aligned} p_k \rho_k &= (I \otimes B_k)\rho(I \otimes B_k) \\ &= \frac{1}{4}(I \otimes V)(I \otimes \Pi_k)[I + r\sigma_3 \otimes I + sI \otimes V^\dagger \sigma_3 V^\dagger \\ &\quad + \sum_{j=1}^3 c_j \sigma_j \otimes (V^\dagger \sigma_j V)](I \otimes \Pi_k)(I \otimes V^\dagger). \end{aligned} \quad (12)$$

Using the relations [26]

$$V^\dagger \sigma_1 V = (t^2 + y_1^2 - y_2^2 - y_3^2)\sigma_1 + 2(ty_3 + y_1 y_2)\sigma_2 + 2(-ty_2 + y_1 y_3)\sigma_3, \quad (13)$$

$$V^\dagger \sigma_2 V = 2(-ty_3 + y_1 y_2)\sigma_1 + (t^2 + y_2^2 - y_1^2 - y_3^2)\sigma_2 + 2(ty_1 + y_2 y_3)\sigma_3, \quad (14)$$

$$V^\dagger \sigma_3 V = 2(ty_2 + y_1 y_3)\sigma_1 + 2(-ty_1 + y_2 y_3)\sigma_2 + (t^2 + y_3^2 - y_1^2 - y_2^2)\sigma_3, \quad (15)$$

and

$$\Pi_0 \sigma_3 \Pi_0 = \Pi_0, \Pi_1 \sigma_3 \Pi_1 = -\Pi_1, \Pi_j \sigma_k \Pi_j = 0, \quad (16)$$

for  $j = 0, 1, k = 1, 2$ , we obtain

$$p_0 \rho_0 = \frac{1}{4}[I + sz_3 I + c_1 z_1 \sigma_1 + c_2 z_2 \sigma_2 + (r + c_3 z_3)\sigma_3] \otimes (V\Pi_0 V^\dagger), \quad (17)$$

$$p_1 \rho_1 = \frac{1}{4}[I - sz_3 I - c_1 z_1 \sigma_1 - c_2 z_2 \sigma_2 + (r - c_3 z_3)\sigma_3] \otimes (V\Pi_1 V^\dagger), \quad (18)$$

where

$$\begin{aligned} z_1 &= 2(-ty_2 + y_1 y_3), \quad z_2 = 2(ty_1 + y_2 y_3), \\ z_3 &= t^2 + y_3^2 - y_1^2 - y_2^2. \end{aligned} \quad (19)$$

Then, we will evaluate the eigenvalues of  $\sum_k \Pi_k \rho^{ab} \Pi_k$  by  $\sum_k \Pi_k \rho^{ab} \Pi_k = p_0 \rho_0 + p_1 \rho_1$ , and

$$\begin{aligned} p_0 \rho_0 + p_1 \rho_1 &= \frac{1}{4}(I + r\sigma_3) \otimes I \\ &\quad + \frac{1}{4}(sz_3 I + c_1 z_1 \sigma_1 + c_2 z_2 \sigma_2 + c_3 z_3 \sigma_3) \otimes V\sigma_3 V^\dagger. \end{aligned} \quad (20)$$

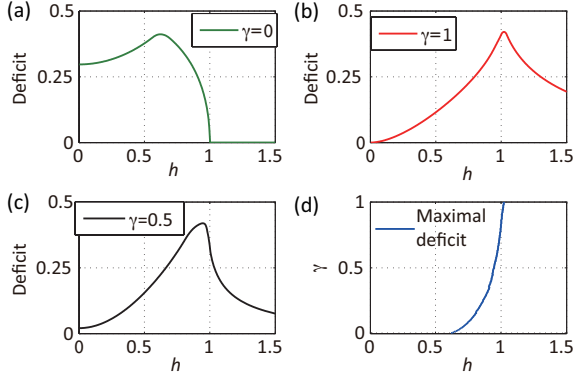


FIG. 2: (Color online) One-way deficit of two adjacent spins in the bulk for the XY model: (a-c) one-way deficit of XY model for  $\gamma \rightarrow 0$  (XX model),  $\gamma = 1$  (transverse field Ising model) and  $\gamma = 0.5$  respectively; (d) Maximal one-way deficit of the XY model.

The eigenvalues of  $p_0\rho_0 + p_1\rho_1$  are the same with the eigenvalues of the states  $(I \otimes V^\dagger)(p_0\rho_0 + p_1\rho_1)(I \otimes V)$ , and

$$\begin{aligned} & (I \otimes V^\dagger)(p_0\rho_0 + p_1\rho_1)(I \otimes V) \\ &= \frac{1}{4}(I + r\sigma_3) \otimes I \\ &+ \frac{1}{4}(sz_3I + c_1z_1\sigma_1 + c_2z_2\sigma_2 + c_3z_3\sigma_3) \otimes \sigma_3. \end{aligned} \quad (21)$$

The eigenvalues of the states in the equation (21) are

$$w_{1,2} = \frac{1}{4} \left( 1 - sz_3 \pm \sqrt{(r - c_3z_3)^2 + c_1^2z_1^2 + c_2^2z_2^2} \right) \quad (22)$$

$$w_{3,4} = \frac{1}{4} \left( 1 + sz_3 \pm \sqrt{(r + c_3z_3)^2 + c_1^2z_1^2 + c_2^2z_2^2} \right). \quad (23)$$

The entropy of  $\sum_k \Pi_k \rho^{ab} \Pi_k$  is  $S(\sum_k \Pi_k \rho^{ab} \Pi_k) = -\sum_{i=1}^4 w_i \log w_i$ . When  $\gamma, h$  are fixed,  $r, s, c_1, c_2, c_3$  is constant. By using  $z_1^2 + z_2^2 + z_3^2 = 1$ , it converts the problem about  $\min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k)$  to the problem about the function of three variables  $z_1, z_2, z_3$  for minimum, that is

$$\min_{\{\Pi_k\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k) = \min_{\{z_1^2 + z_2^2 + z_3^2 = 1\}} S(\sum_k \Pi_k \rho^{ab} \Pi_k). \quad (24)$$

By Eqs. (1), (6), (24), the one-way deficit of the X states in Eq. (3) is given by

$$\begin{aligned} & \Delta^\rightarrow(\rho^{ab}) \\ &= \min_{\{z_1^2 + z_2^2 + z_3^2 = 1\}} \left( -\sum_{i=1}^4 w_i \log w_i \right) + \sum_{i=1}^4 \lambda_i \log \lambda_i. \end{aligned} \quad (25)$$

Next, we use the equation above to calculate the one-way deficit of two adjacent spins in the bulk for the XY model and analyze the in different phases..

The main results are shown in Fig. 1, in which we plot the one-way deficit of two adjacent spins in the bulk

of XY model as a function of  $h, \gamma$ . When  $\gamma$  is a fixed value, we observe that as the transverse field strength  $h$  increases, the one-way deficit increases for small  $h$  and decreases for large  $h$ , see Fig. 2 (c) for  $\gamma = 0.5$ . When  $\gamma \rightarrow 1$ , maximum of the one-way deficit is attained near  $h = 1$ , see Fig. 2 (d).

In Fig. 2 (a) for the case that  $\gamma \rightarrow 0$ , our model reduces to XX model. We find that the one-way deficit is nonzero in the domain  $h \in [0, 1]$  and then suddenly becomes zero when  $h \geq 1$ . As the XX model undergoes a first order transition at the critical point  $h = 1$  from fully polarized to a critical phase with quasi-long-range order, we conclude that one-way deficit can be used to detect quantum phase of the XX model. The conclusion is in consistent with the result obtained in [27].

In Fig. 2 (b) for the case that  $\gamma = 1$ , the model reduces to transverse field Ising model. We find that one-way deficit of Ising model increases for small  $h$  and decreases for large  $h$ . When one-way deficit achieve the maximum nearly at  $h = 1$ , transverse field Ising model undergoes a first order transition. We infer that one-way deficit can also be used to detect quantum phase of the transverse field Ising model.

### III. CONCLUSION

We have given a method to analytically evaluate the one-way deficit of the thermal ground states of two adjacent spins in the bulk for the XY model in the thermodynamic limit. We have drawn the diagram of one-way deficit of the XY model. We find that we can use one-way deficit to detect quantum phase of the XX model and transverse field Ising model. We find the sudden death of the one-way deficit in XX model as the strength of transverse field increases to  $h = 1$ . For the transverse field Ising model, the one-way deficit of the thermal ground state of nearby two spins arrives its maximal value at the critical point  $h = 1$ . On one hand, our results may shed lights on the study of properties of quantum correlations in different quantum phases of many body systems. On the other hand, our investigations will also benefit many applications including quantum phase transition detection and the capacity of quantum computation evaluation in critical systems. Also numerical techniques such as DMRG, MPS and exact diagonalization methods deserve to be used to investigate extensive problems of quantum phase transitions from the perspective of quantum information and quantum correlations, including finite-temperature phase transitions, phase transitions of other quantum many-body models or even topological phase transitions.

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- [1] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).
  - [2] M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Phys. Rev. Lett. **90**, 100402 (2003).
  - [3] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
  - [4] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak, Phys. Rev. A **71**, 062307 (2005).
  - [5] M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. A **67**, 062104 (2003).
  - [6] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **108**, 250501 (2012).
  - [7] T. K. Chuan, J. Maillard, K. Modi, T. Paterek, M. Paternostro, and M. Piani, Phys. Rev. Lett. **109**, 070501 (2012).
  - [8] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Nature (London) **464**, 45 (2010).
  - [9] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
  - [10] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
  - [11] L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004).
  - [12] R. Orús and T. C. Wei, Phys. Rev. B **82**, 155120 (2010).
  - [13] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
  - [14] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).
  - [15] R. Dillenschneider, Phys. Rev. B **78**, 224413 (2008).
  - [16] J. Cui, J. P. Cao, and H. Fan, Phys. Rev. A **85**, 022338 (2012).
  - [17] J. Cui, M. Gu, L. C. Kwek, M. F. Santos, H. Fan, and V. Vedral, Nat Commun **3**, 812 (2012).
  - [18] J. Cui, L. Amico, H. Fan, M. Gu, A. Hamma, and V. Vedral, Phys. Rev. B **88**, 125117 (2013).
  - [19] F. Franchini, J. Cui, L. Amico, H. Fan, M. Gu, V. Korepin, L. C. Kwek, and V. Vedral, Phys. Rev. X **4**, 041028 (2014).
  - [20] S. L. Liu, Q. Quan, J. J. Chen, Y. R. Zhang, W. L. Yang, and H. Fan, Sci. Rep. **6**, 29175 (2016).
  - [21] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **106**, 160401 (2011).
  - [22] V. Mukherjee, U. Divakaran, A. Dutta, and D. Sen, Phys. Rev. B **76**, 174303 (2007); S. Garnerone, N. T. Jacobson, S. Haas, and P. Zanardi, Phys. Rev. Lett. **102**, 057205 (2009); F. Franchini, A. R. Its, and V. E. Korepin, J. Phys. A **41**, 025302 (2008).
  - [23] S. Y. Liu, Y. R. Zhang, W. L. Yang, and H. Fan, Ann. Phys **362**, 805. (2015).
  - [24] P. Pfeuty, Ann. Phys. **57**, 79 (1970).
  - [25] W. Son, L. Amico, F. Plastina and V. Vedral, Phys. Rev. A **79**, 022302 (2009).
  - [26] S. Luo, Phys. Rev. A **77**, 042303 (2008).
  - [27] Y. K. Wang, arXiv:1601.02750.